

Minimizing Higgs Potentials via Numerical Polynomial Homotopy Continuation

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The study of models with extended Higgs sectors requires to minimize the corresponding Higgs potentials, which is in general very difficult. Here, we apply a recently developed method, called numerical polynomial homotopy continuation (NPHC), which guarantees to find *all* the stationary points of the Higgs potentials with polynomial-like nonlinearity. The detection of all stationary points reveals the structure of the potential with maxima, metastable minima, saddle points besides the global minimum. We apply the NPHC method to the most general Higgs potential having two complex Higgs-boson doublets and up to five real Higgs-boson singlets. Moreover the method is applicable to even more involved potentials. Hence the NPHC method allows to go far beyond the limits of the Gröbner basis approach.

I. INTRODUCTION

The Standard Model (SM) comes with a simple Higgs potential: due to gauge invariance and renormalizability, there is only one Higgs doublet which appears in the potential in one bilinear and one quartic term. In particular, finding the global minimum of the Higgs potential, which gives the vacuum-expectation value of the Higgs field at the stable vacuum, is quite straightforward.

The situation is very different in models with extended Higgs sectors. For instance in the general two-Higgs-doublet model (THDM), introduced by T.D. Lee [1] in order to allow for CP violation in the Higgs sector, we encounter already three bilinear terms as well as seven quartic terms in the potential, corresponding to 14 real parameters. For some recent works on the THDM we refer to [2–6]. Much motivation for the two-Higgs doublet model arises from supersymmetry, since supersymmetric extensions of the SM require to have at least two Higgs doublets – besides the corresponding superpartners. The minimal supersymmetric extension of the Standard Model (MSSM) has exactly two Higgs doublets and is therefore a special THDM; for a review see [7]. However the MSSM has a number of drawbacks, in particular the so-called μ parameter in the superpotential has to be adjusted by hand to the electroweak scale, that is, the electroweak scale appears *not* via electroweak symmetry breaking. This μ problem may be circumvented by a vacuum-expectation value of an additional (complex) Higgs-boson singlet, leading to the next-to-minimal supersymmetric standard model (NMSSM); for reviews see [8, 9]. Different supersymmetric models have been proposed which have two Higgs-boson doublets and one or several Higgs-boson singlets; an overview can be found in [10].

Other examples of extended Higgs sectors appear in the study of additional discrete symmetries. Extending the Higgs sector and assigning all fermions and scalars to certain irreducible representations with respect to the discrete symmetry (for instance the quaternion group with eight elements, Q_8), the masses and mixing angles of the quarks and leptons may be given by very few parameters; see for instance [11, 12].

While dealing with an extended Higgs sector it is mandatory to minimize the Higgs potential. The determination of the global minimum is essential to determine the masses of the fermions via the Yukawa couplings and to ensure that the electroweak symmetry breaking behavior follows $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$. However, finding the global minimum of a given non-linear multivariate function is a highly non-trivial task. There are a number of numerical methods, e.g., conjugate gradient, steepest descent, simulated annealing, genetic algorithm, etc. which are devised to find the global minimum but for one reason or another almost all of them are known to fail even for a relatively small system: the multivariate functions usually come with many minima and the above mentioned methods can be easily stuck to a local minimum instead of the global minimum. Hence, one can never be sure that the actual global minimum is found by these methods.

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A more systematic approach to minimize the potential is given by the solution of the stationarity equations. Suppose the potential is bounded from below, then the stationary points with the lowest potential value are the global minima, that is, the vacua. However, again, solving the stationary equations amounts to finding the solutions of a system of non-linear multivariate equations which is equally hard. However, the finding of all stationary points not only reveals the global minimum, but gives some valuable insight to the structure of the potential: in this way all maxima, saddle points, local minima besides the global minima are detected.

If the potential and hence the system of stationary equations have polynomial-like non-linearity (as opposed to involving transcendental functions etc.), then one viable option is to use a symbolic method called the Gröbner basis approach. The Gröbner basis approach was used to find all the stationary points and hence the global minimum of extended Higgs potentials in [13]. There, for a special case with two doublets and two real singlets, it was shown that extended Higgs potentials may develop a very rich structure of stationary points.

The Gröbner basis approach, however, has several severe drawbacks: firstly, the Buchberger algorithm is known to suffer from the exponential space complexity, i.e., the memory (RAM) required by the machine exponentially blows up with increasing number of variables, number of equations, degree of equations or number of monomials. The Buchberger algorithm is highly sequential and hence very difficult to parallelize. Thus, practically, the Gröbner basis approach is limited to only small polynomial systems of equations of lower degrees.

In particular, in the example of two doublets and n real singlets it is currently not possible to go far beyond the case $n = 2$, as considered in [13].

In this work we shall introduce the numerical polynomial homotopy continuation method [14, 15] as a method to find all stationary points of extended Higgs potentials. We shall demonstrate that this method is very powerful and overcomes all of the above mentioned shortcomings of the Gröbner basis approach. The approach allows to identify *all* stationary solutions and supposed the potential is bounded from below the global minima are therefore certainly detected. As an example we minimize the most general potential with two complex Higgs-boson doublets and n real Higgs-boson singlets. We do a detailed analysis for the cases up to $n = 5$ in this paper, but we emphasize that we can straightforwardly go up to $n = 10$ even on a regular desktop machine. And by exploiting the parallelizable nature of the method, it is possible to go even beyond the $n = 10$ case.

II. THE HIGGS POTENTIAL AND THE STATIONARITY EQUATIONS

In this Section, besides fixing the notation we explicitly write down the potential and its stationary equations along with the tadpole conditions.

Let us consider a Higgs potential with two Higgs-boson doublets and n Higgs-boson singlets [13]. By convention we assume that both doublets carry hypercharge $y = +1/2$, and denote the complex doublet fields by

$$\varphi_i(x) = \begin{pmatrix} \varphi_i^+(x) \\ \varphi_i^0(x) \end{pmatrix}, \quad i = 1, 2. \quad (1)$$

For the singlets we assume real fields which we denote by

$$\phi_i(x), \quad i = 1, \dots, n. \quad (2)$$

Of course any complex singlet field may be decomposed into two real singlet fields.

We employ the bilinear approach for the Higgs doublets (see [16] for details). The Higgs-boson doublets can only appear in the form of scalar products $\varphi_i^\dagger \varphi_j$ with $i, j \in \{1, 2\}$ in the potential, ensuring gauge invariance. As shown in [16] we may replace the scalar products by the bilinears K_0, K_1, K_2 and K_3 ,

$$\varphi_1^\dagger \varphi_1 = (K_0 + K_3)/2, \quad \varphi_1^\dagger \varphi_2 = (K_1 + iK_2)/2, \quad \varphi_2^\dagger \varphi_2 = (K_0 - K_3)/2, \quad \varphi_2^\dagger \varphi_1 = (K_1 - iK_2)/2. \quad (3)$$

The bilinears have to fulfill the conditions

$$K_0 \geq 0, \quad K_0^2 - K_1^2 - K_2^2 - K_3^2 \geq 0. \quad (4)$$

As shown in [16] the four quantities K_α , satisfying (4), parameterize the gauge orbits of the Higgs doublets. The advantage of the replacement of the Higgs-boson doublets by the bilinears is that we eliminate the $SU(2)_L \times U(1)_Y$ gauge degrees of freedom and moreover simplify the potential since the K_α are bilinear in the Higgs-boson doublets. We thus end up with a potential of the form $V(K_0, K_1, K_2, K_3, \phi_1, \dots, \phi_n)$.

A. Stationary points

The most general Higgs potential of two Higgs doublets and n real singlets reads

$$V(K_0, K_1, K_2, K_3, \phi_1, \dots, \phi_n) = \xi_\alpha K_\alpha + \eta_{\alpha\beta} K_\alpha K_\beta + a_i \phi_i + b_{ij} \phi_i \phi_j + c_{ijk} \phi_i \phi_j \phi_k + d_{ijkl} \phi_i \phi_j \phi_k \phi_l + e_{i\alpha} \phi_i K_\alpha + f_{ij\alpha} \phi_i \phi_j K_\alpha \quad (5)$$

with $\alpha, \beta \in \{0, \dots, 3\}$ and $i, j, k, l \in \{1, \dots, n\}$ and summation convention for repeated indices. Any additional term violates $SU(2)_L \times U(1)_Y$ invariance respectively renormalizability. Note that the coefficients $\eta_{\alpha\beta}$, b_{ij} , c_{ijk} , d_{ijkl} , and $f_{ij\alpha}$ are symmetric in their greek respectively latin indices. Depending on the number of real singlets n we count $14 + 5n + 5\binom{n+1}{2} + \binom{n+2}{3} + \binom{n+3}{4}$ coefficients in the most general case. Hence for the case $n = 5$ we have 219 coefficients in the general case.

In order to determine the stationary points of the Higgs potential (5) we take the constraint (4) into account. We distinguish the possible cases of stationary points with respect to the $SU(2)_L \times U(1)_Y$ symmetry breaking behaviour [13, 16]:

- Unbroken electroweak gauge symmetry: This is a stationary point with

$$K_0 = K_1 = K_2 = K_3 = 0. \quad (6)$$

A global minimum of this type has vanishing vacuum expectation values for the doublet fields (1). The stationary points of this type are found for vanishing Higgs-doublet fields, corresponding to vanishing bilinears K_α and requiring a vanishing gradient with respect to the remaining real singlets:

$$\nabla V(\phi_1, \dots, \phi_n) = 0. \quad (7)$$

- Fully broken electroweak gauge symmetry: This is a stationary point with

$$K_0 > 0, \quad K_0^2 - K_1^2 - K_2^2 - K_3^2 > 0. \quad (8)$$

A global minimum of this type has non-vanishing vacuum-expectation values for the charged components of the doublets fields in (1), thus gives fully broken $SU(2)_L \times U(1)_Y$; see [16]. The stationary points of this type are found by requiring a vanishing gradient with respect to all bilinears K_α and all singlet fields:

$$\nabla V(K_0, K_1, K_2, K_3, \phi_1, \dots, \phi_n) = 0. \quad (9)$$

In this case the constraints (8) on the bilinears must be checked explicitly for the real solutions.

- Partially broken electroweak gauge symmetry: This is a stationary point with

$$K_0 > 0, \quad K_0^2 - K_1^2 - K_2^2 - K_3^2 = 0. \quad (10)$$

For a global minimum of this type follows the desired partial breaking of $SU(2)_L \times U(1)_Y$ down to $U(1)_{em}$; see [16]. Using the Lagrange multiplier method, these stationary points are given by the real solutions of the system of equations

$$\begin{aligned} \nabla [V(K_0, K_1, K_2, K_3, \phi_1, \dots, \phi_n) - u \cdot (K_0^2 - K_1^2 - K_2^2 - K_3^2)] &= 0, \\ K_0^2 - K_1^2 - K_2^2 - K_3^2 &= 0, \end{aligned} \quad (11)$$

where u is a Lagrange multiplier. The inequality in (10) must be checked explicitly for the real solutions found for (11).

If the potential is bounded from below, the global minima will be among the stationary solutions. Plugging the solutions into the potential the global minima are the solutions with the lowest potential value.

The systems of stationary equations, (7), (9), (11) are non-linear, multivariate, inhomogeneous systems of polynomial equations of degree 3. For the case of two doublets and n real singlets we have for the unbroken, fully broken, and the partially broken systems sets with n , $4 + n$, and $5 + n$ equations, respectively. The number of indeterminants equals the number of equations.

III. NUMERICAL POLYNOMIAL HOMOTOPY CONTINUATION METHOD

Finding the global minima or all stationary points of a given potential is usually a very difficult problem. As outlined in the Introduction, a systematic approach is to solve the stationarity systems of equations. While the problem of finding all solutions which include minima, maxima and saddle points, compared to only finding minima, seems more complicated at first sight, a powerful method called the Gröbner basis method can be used to solve the stationary equations as long as they are polynomial equations. However, due to the algorithmic complexity issues discussed in the Introduction, we are practically restricted to solve only small systems. In particular, going beyond the NMSSM, involving two Higgs-boson doublets and one complex singlet (equivalent to two real singlets), treated in [13], appears to be difficult to handle with the Gröbner basis approach.

Here, we present a novel numerical method, the numerical polynomial homotopy continuation (NPHC) method, which overcomes all the shortcomings of the Gröbner basis method. The NPHC method was recently introduced in particle theory and statistical mechanics areas in [17] to find all the so-called Gribov copies of the Landau gauge-fixing conditions on the lattice [18–20] and subsequently found many applications in many areas of theoretical physics including string phenomenology, lattice field theories, theoretical chemistry, non-linear dynamics, etc. [21–25].

The basic strategy behind numerical continuation methods is: first find the solutions of a simple system of equations which shares several important features with the given system. Then, in a second step, starting from each of these solutions one continues them towards the given system in a systematic way. Homotopy continuation methods have been around already for several decades [26, 27], but with more recent machinery like the NPHC method used in the present article, the method is guaranteed to find all isolated solutions of systems of polynomial equations [14, 15].

Suppose we want to solve a system of m polynomial equations

$$P(x) = \begin{pmatrix} p_1(x) \\ \vdots \\ p_m(x) \end{pmatrix} = 0 \quad (12)$$

in the variables $x = (x_1, \dots, x_m)^T$, which is known to have only isolated solutions. Then the Bézout's Theorem asserts that a system of m polynomial equations in m variables has at most $\prod_{i=1}^m d_i$ isolated complex (which obviously include real) solutions where d_i is the degree of the i th polynomial. This upper bound is called the *classical Bézout bound*, and it is known to be sharp for generic systems (i.e., for generic values of the coefficients of the polynomials $p_i(x)$).

Then, we formulate a homotopy (or, a set of parametric equations) as

$$H(x, t) = P(x)(1 - t) + \gamma t Q(x), \quad (13)$$

where γ is a complex number and

$$Q(x) = \begin{pmatrix} q_1(x) \\ \vdots \\ q_m(x) \end{pmatrix} = 0 \quad (14)$$

is again a system of m polynomial equations. Now, varying the parameter $t \in [0, 1]$, H can be deformed from the *start system* $H(x, 1) = \gamma Q(x)$ at $t = 1$ into the polynomial system we want to solve, $H(x, 0) = P(x)$ at $t = 0$. The following conditions have to be satisfied in order to guarantee that all solutions of P can be computed from this homotopy:

1. The solutions of $Q(x) = 0$ can be computed relatively straightforwardly.
2. The number of solutions of $Q(x) = 0$ satisfies the classical Bézout bound for $P(x) = 0$ as an equality.
3. The solution set of $H(x, t) = 0$ for $t \in (0, 1]$ consists of a finite number of smooth paths, called *homotopy paths*, which are parameterized by t .
4. Every isolated solution of $H(x, 0) = P(x) = 0$, including the multiple roots, can be reached by some path originating at a solution of $H(x, 1) = \gamma Q(x) = 0$.

Satisfying the first two criteria hinges on a suitable choice of the start system Q . Criteria 3 and 4 are guaranteed to be satisfied, for generic constants γ in (13) [14].

The start system $Q(x) = 0$ can, for example, be taken to be

$$Q(x) = \begin{pmatrix} x_1^{d_1} - 1 \\ \vdots \\ x_m^{d_m} - 1 \end{pmatrix} = 0, \quad (15)$$

where d_i is the degree of the i^{th} polynomial of $P(x) = 0$. The system (15) is easy to solve and guarantees that the total number of start solutions is $\prod_{i=1}^m d_i$ and all solutions are nonsingular.

Each homotopy path, starting at a solution of $Q(x) = 0$ at $t = 1$, is tracked to $t = 0$ using a path tracking algorithm, e.g., Euler predictor and Newton corrector methods. There are a number of freeware packages well-equipped with path trackers such as PHCpack [28], HOM4PS2 [29], and Bertini [30]. We have used Bertini and HOM4PS2 to obtain the results in this paper. Tracking the solutions to $t = 0$, the set of endpoints of these homotopy paths is the set of all solutions to $P(x) = 0$. Since each homotopy path can be tracked independently, NPHC is inherently parallelizable. The NPHC method does not suffer from any major computational complexity. It takes floating point coefficients in very naturally.

The set of real solutions can be obtained from the set of complex solutions by considering the imaginary part of the solutions (typically, up to a numerical tolerance). We remark that the approach of [31] implemented in alphaCertified [32] can be used to certify the reality or non-reality of a nonsingular solution given a numerical approximation of the solution. The ability to compute all complex solutions, and thus all real solutions, distinguishes the NPHC method from most other methods.

IV. MINIMIZATION OF THE HIGGS POTENTIAL

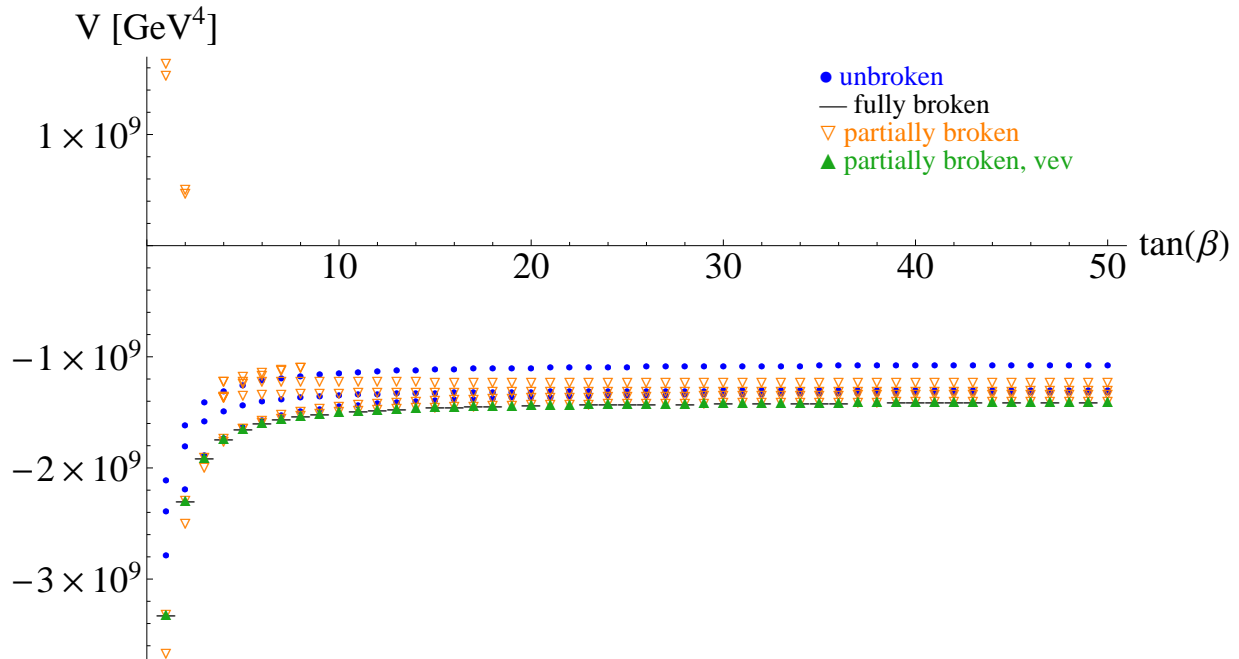


Figure 1: Higgs potential values for the stationary points with 2 Higgs-boson doublets and 2 Higgs-boson singlets. The parameter $\tan(\beta)$ is varied in the range $\tan(\beta) = 1, \dots, 50$. All other parameters are fixed as described in the text. The different types of stationary points are shown with respect to the electroweak symmetry breaking behavior. The filled dots denote the unbroken, the empty squares the fully broken solutions and the triangles the partially broken solutions, respectively. Only the full green triangles correspond to stationary points with vevs as given in (18). The stationary point with the lowest potential value is the global minimum.

We now want to apply the NPHC method to the minimization problem of the Higgs potential (5), i.e., we want to solve the stationarity systems of equations (7), (9), and (11). We note that the solution with the correct electroweak symmetry breaking has to come from the set (11). We start with numerically fixing the vacuum-expectation values (vevs) v , $\tan(\beta)$ and v'_i , with $i = 1, \dots, n$ with n being the number of real singlets. As usual $\tan(\beta) = v_2/v_1$ denotes the ratio of the vevs of the two Higgs-boson doublets, $v^2 = v_1^2 + v_2^2$ and v'_i are the vevs of the real singlets:

$$\langle \varphi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \varphi_2 \rangle = \frac{1}{\sqrt{2}} e^{i\theta} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \quad \langle \phi_i \rangle = \frac{1}{\sqrt{2}} v'_i \quad (16)$$

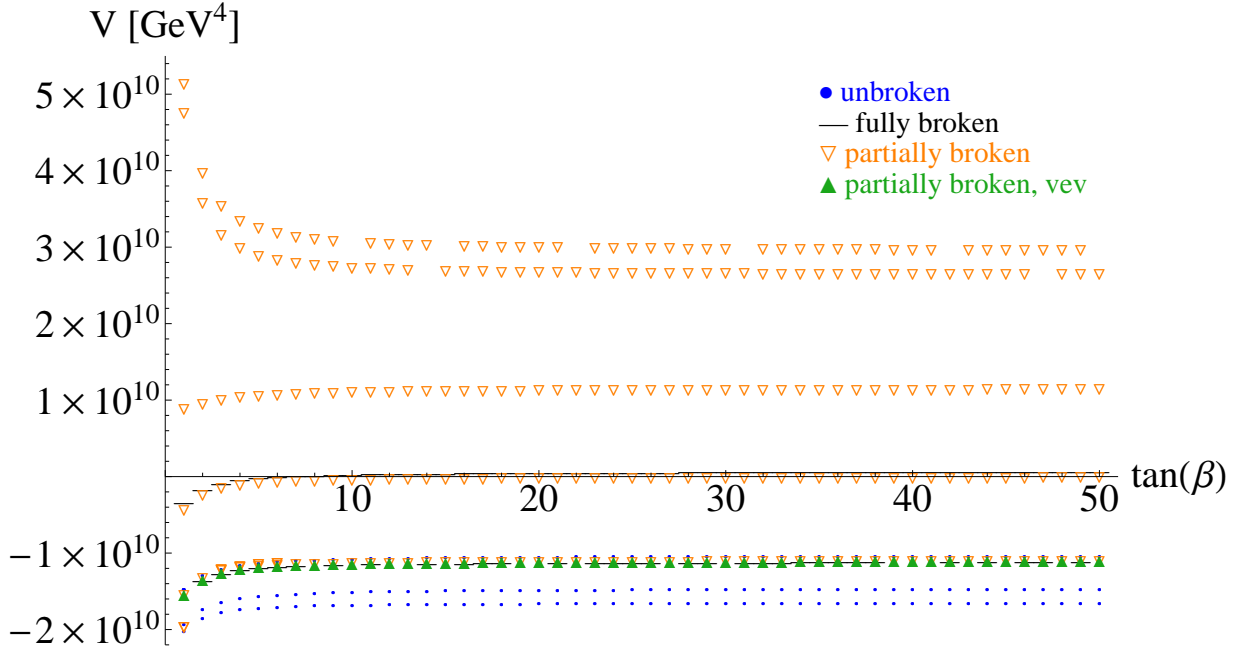


Figure 2: Similar to Fig. 1 the Higgs potential values for the stationary points with two Higgs-boson doublets but with 3 Higgs-boson singlets.

Except for ξ_α , $\alpha = 0, \dots, 3$ and a_i , $i = 1, \dots, n$, we first fix all coefficients in (5) numerically. The parameters ξ_α and a_i are then fixed by employing the tadpole conditions,

$$\langle \nabla V(K_0, K_1, K_2, K_3, \phi_1, \dots, \phi_n) \rangle = 0. \quad (17)$$

In this way we gain solutions corresponding to the correct electroweak symmetry breaking behavior with the desired vevs.

For the case considered here with two Higgs-boson doublets and up to five real singlets we choose the vevs

$$v = 246 \text{ GeV}, \quad v'_1 = 100 \text{ GeV}, \quad v'_2 = 150 \text{ GeV}, \quad v'_3 = 200 \text{ GeV}, \quad v'_4 = 250 \text{ GeV}, \quad v'_5 = 300 \text{ GeV}, \quad (18)$$

a vanishing phase θ of the second doublet and vary the ratio of the doublet vevs in the range $\tan(\beta) = 1, \dots, 50$. All coefficients in (5), except for ξ_α and a_i , that is, $\eta_{00}, \eta_{01}, \dots, f_{nn3}$ are fixed in the order they appear in the potential. As a numerical example we assign the values 0.1, 0.2, 0.3, 0.4, 0.5, 0.1, \dots , 0.5, 0.1 \dots in turn. Note the symmetry of the coefficients, that is, only ordered indices appear, for instance b_{21} is replaced by b_{12} in the implicit summation in (5) before the values are assigned.

With all parameters numerically fixed we solve the systems of equations (7), (9), and (11). As discussed in Sec. III we employ the NPHC method for the solution of the stationary equations. We note that on a regular desktop machine we could solve the systems for $n = 10$ in around 16 hours for a fixed set of parameters. However, since we want to scan over a large range of parameter values, due to the available limited computer resources, we restrict ourselves to smaller systems, i.e., $n = 2, \dots, 5$. The largest of these systems can be solved in around 4 hours for a given set of parameters on the desktop machine.

For two Higgs-boson doublets and $n = 2, \dots, 5$ real singlets we show in Table I the number of complex stationary solutions with respect to the fully, unbroken and partially broken cases.

From all complex solutions the real solutions are sorted out (practically we require all imaginary parts to be smaller than 10^{-8}). The number of real solutions is typically much smaller and depends on the chosen parameters. The fact that the number of complex solutions is constant with varied parameters is a good indication that no solutions are missed. For the fully broken set subsequently the conditions (8) are checked. Similarly, for the partially broken case, the solutions have to fulfill (10).

The potential values for all stationary solutions passing all checks are plotted in Figs. 1, 2, 3, 4 for 2, 3, 4, 5 real singlets, respectively. The parameter $\tan(\beta)$ is varied in the range $\tan(\beta) = 1, \dots, 50$. Since the potential is bounded

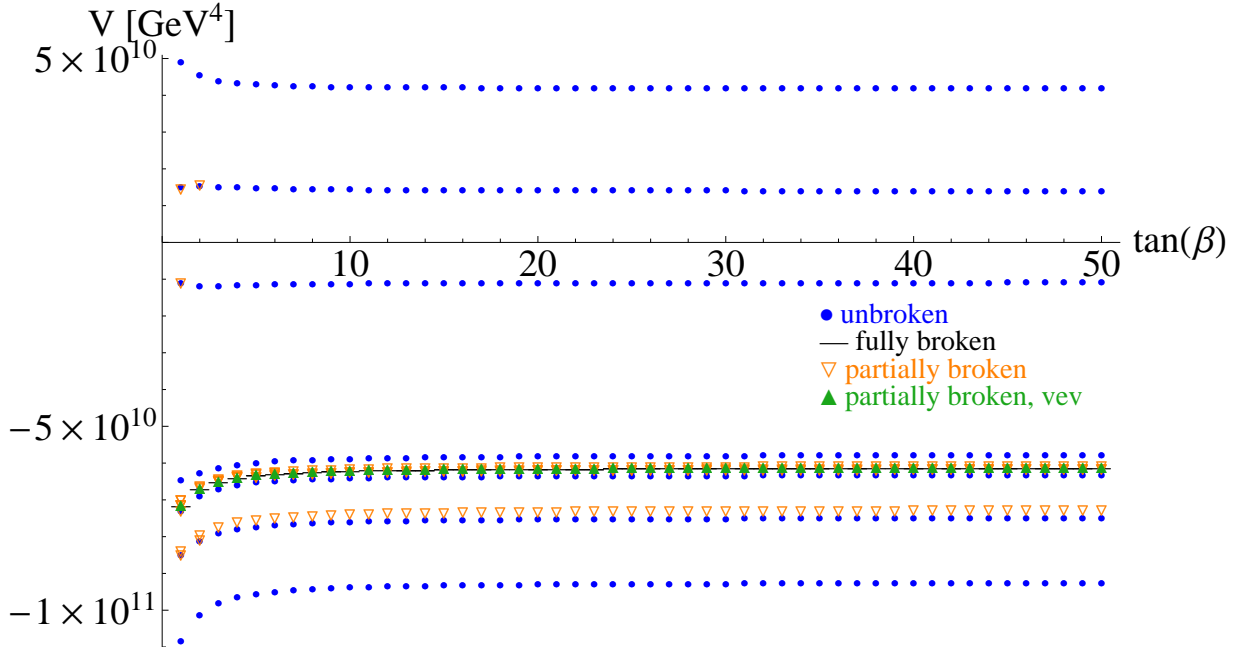


Figure 3: Similar to Fig. 1 the Higgs potential values for the stationary points with two Higgs-boson doublets but with 4 Higgs-boson singlets.

n	unbroken	fully broken	partially broken
2	9	9	54
3	27	27	162
4	75	81	486
5	225	243	1458

Table I: Number of complex solutions found for the most general Higgs potential with two doublets and n singlets. The number of solutions are given with respect to the systems of equations (7) (unbroken case), (9) (fully broken case), and (11) (partially broken case), respectively.

from below the stationary solutions with the lowest potential value are the global minima. The stationary solutions are denoted by circles, squares, and triangles corresponding to unbroken, fully broken, and partially broken solutions, respectively. In case of the partially broken solutions empty triangles show solutions which do not give the required vevs (18), whereas the full triangles show the solutions which yield to the required vevs (with a precision of 8 digits).

In order to check whether the partially broken solutions give the desired vevs (18) we calculate the vevs from the solutions via (16), that is,

$$\sqrt{2K_0} = v, \quad \sqrt{\frac{K_0 - K_3}{K_0 + K_3}} = \tan(\beta), \quad \sqrt{2} \phi_i = v'_i. \quad (19)$$

We find a very rich structure of stationary points. Obviously the global minimum, that is the vacuum, not always correspond to the physically desired solution. As can be seen in Fig. 1 in the case of two real singlets, the global minimum breaks electroweak gauge symmetry partially, however yields undesired vevs for small $\tan(\beta)$ values.

Let us note that quite generically we find nearly degenerate potential values for different stationary solutions. Note that even when the potential values are nearly degenerate the solutions may be located at very different field values. For instance in the case of two Higgs-boson doublets and two singlets we find for $\tan(\beta) = 10$ the two deepest stationary solutions as shown explicitly in Tab. II. In this numerical example only the solution 1 corresponds to the desired vevs (18), however solution 2 is the global minimum. This behavior of nearly degenerate vacua was already mentioned for the case of the NMSSM in [13] which we confirm for more general cases.

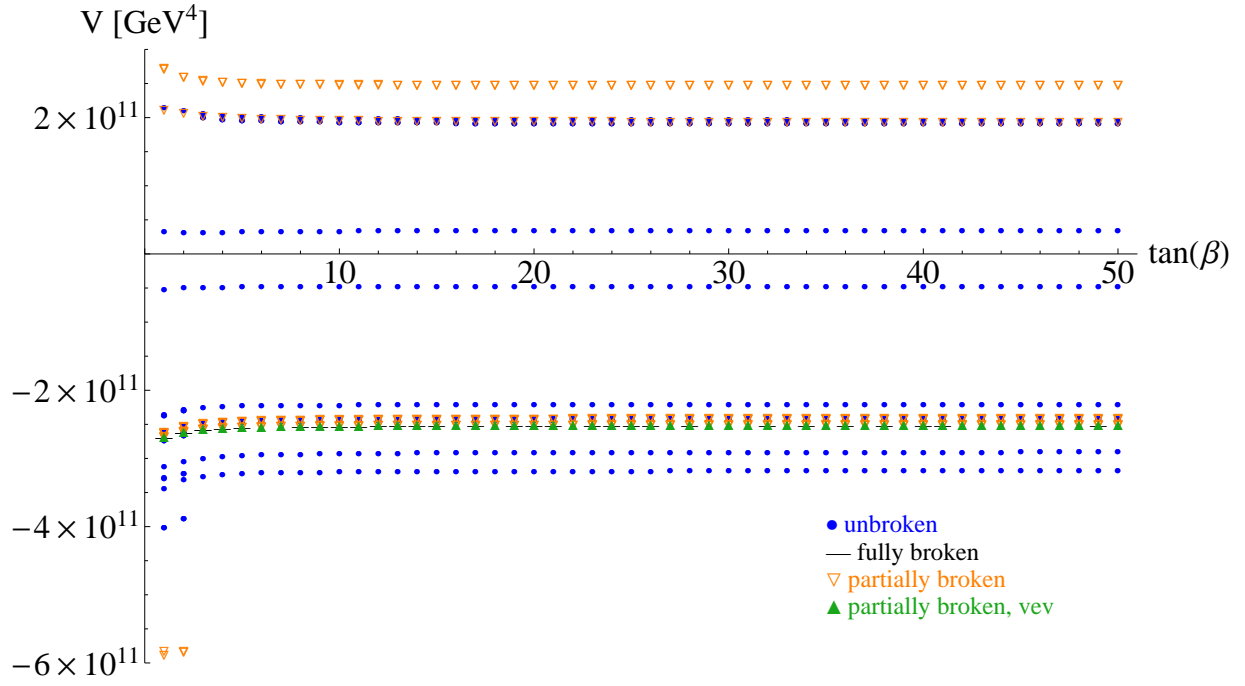


Figure 4: Similar to Fig. 1 the Higgs potential values for the stationary points with two Higgs-boson doublets but with 5 Higgs-boson singlets.

	K_0	K_1	K_2	K_3	ϕ_1	ϕ_2	u	V
solution 1:	30257	5991.6	0.0046390	-29658	70.710	106.06	$-1.0334 \cdot 10^{-8}$	$-1.5045 \cdot 10^9$
solution 2:	34878	8085.9	-4701.0	-33600	55.630	123.54	-0.0036596	$-1.5051 \cdot 10^9$

Table II: The two deepest solutions explicitly for the case of two singlets and $\tan(\beta) = 10$. Solution 1 corresponds to the desired vevs, whereas solution 2 gives wrong vevs.

V. CONCLUSIONS

The extended Higgs models have attracted a lot of attention recently, in particular with the on-going experiments at the Large Hadron Collider. The main technical difficulty in studying these models is that accurately finding the global minimum of the corresponding Higgs potential is highly non-trivial due to its non-linear nature. Whereas the Higgs potential of the Standard Model is rather simple, this is in general not longer true for extended potentials. The solution of the stationarity equations allows to identify the vacuum, as long as the potential is bounded from below. In this paper, we have applied the numerical polynomial homotopy continuation method to solve the non-linear, multivariate systems of polynomial stationary equations for rather involved systems of Higgs potentials. In contrast to most minimization methods this method guarantees the detection of the global minimum given by the stationary solution with the lowest potential value.

We have shown that the case of the most general Higgs potential with two complex doublets and five real singlets is solvable via numerical polynomial homotopy continuation. The results show a rich structure of stationary solutions with local maxima, saddle points and minima. In this way, we have shown that a large set of parameter values can be excluded from the physically viable regions. Among the solutions we typically find solutions with nearly degenerate potential values, however far away in terms of the Higgs-boson fields. Let us note that the numerical polynomial homotopy continuation method allows to go far beyond the limits of two doublets and five singlets.

Acknowledgments

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- [1] T. Lee, Phys.Rev. **D8**, 1226 (1973).
 - [2] C. Nishi, Phys.Rev. **D77**, 055009 (2008), 0712.4260.
 - [3] I. Ginzburg, I. Ivanov, and K. Kanishev, Phys.Rev. **D81**, 085031 (2010), 0911.2383.
 - [4] P. Ferreira, H. E. Haber, M. Maniatis, O. Nachtmann, and J. P. Silva, Int.J.Mod.Phys. **A26**, 769 (2011), 1010.0935.
 - [5] M. Maniatis and O. Nachtmann, JHEP **1111**, 151 (2011), 1106.1436.
 - [6] G. Branco *et al.*, (2011), 1106.0034.
 - [7] S. P. Martin, (1997), hep-ph/9709356.
 - [8] M. Maniatis, Int.J.Mod.Phys. **A25**, 3505 (2010), 0906.0777.
 - [9] U. Ellwanger, C. Hugonie, and A. M. Teixeira, Phys.Rept. **496**, 1 (2010), 0910.1785.
 - [10] V. Barger, P. Langacker, H.-S. Lee, and G. Shaughnessy, Phys.Rev. **D73**, 115010 (2006), hep-ph/0603247.
 - [11] W. Grimus and L. Lavoura, Phys.Lett. **B572**, 189 (2003), hep-ph/0305046.
 - [12] M. Frigerio, S. Kaneko, E. Ma, and M. Tanimoto, Phys.Rev. **D71**, 011901 (2005), hep-ph/0409187.
 - [13] M. Maniatis, A. von Manteuffel, and O. Nachtmann, Eur.Phys.J. **C49**, 1067 (2007), hep-ph/0608314.
 - [14] A. J. Sommese and C. W. Wampler, *The numerical solution of systems of polynomials arising in Engineering and Science* (World Scientific Publishing Company, 2005).
 - [15] T. Y. Li, Handbook of numerical analysis **XI**, 209 (2003).
 - [16] M. Maniatis, A. von Manteuffel, O. Nachtmann, and F. Nagel, Eur.Phys.J. **C48**, 805 (2006), hep-ph/0605184.
 - [17] D. Mehta, Ph.D. Thesis, The University of Adelaide, Australasian Digital Theses Program (2009).
 - [18] L. von Smekal, D. Mehta, A. Sternbeck, and A. G. Williams, PoS **LAT2007**, 382 (2007), 0710.2410.
 - [19] L. von Smekal, A. Jorkowski, D. Mehta, and A. Sternbeck, PoS **CONFINEMENT8**, 048 (2008), 0812.2992.
 - [20] D. Mehta and M. Kastner, Annals Phys. **326**, 1425 (2011), 1010.5335.
 - [21] D. Mehta, A. Sternbeck, L. von Smekal, and A. G. Williams, PoS **QCD-TNT09**, 025 (2009), 0912.0450.
 - [22] D. Mehta, Phys.Rev. **E84**, 025702 (2011), 1104.5497.
 - [23] D. Mehta, Adv.High Energy Phys. **2011**, 263937 (2011), 1108.1201.
 - [24] M. Kastner and D. Mehta, Phys. Rev. Lett. **107**, 160602 (2011).
 - [25] D. Mehta, M. Kastner, and J. D. Hauenstein, (2012), 1202.3320.
 - [26] B. Roth, Ph.D. Thesis, Columbia University (1962).
 - [27] E. L. Allgower and K. Georg, *Introduction to Numerical Continuation Methods* (John Wiley & Sons, New York, 1979).
 - [28] J. Verschelde, ACM Trans. Math. Soft. **25**, 251 (1999).
 - [29] T. L. Lee, T. Y. Li, and C. H. Tsai, Computing **83**, 109 (2008).
 - [30] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler, Available at www.nd.edu/~sommese/bertini.
 - [31] J. D. Hauenstein and F. Sottile, To appear in ACM Trans. Math. Softw. (2012).
 - [32] J. D. Hauenstein and F. Sottile, Available at www.math.tamu.edu/~sottile/research/stories/alphaCertified.